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The solution to the discrete-time Lyapunov equation

The Lyapunov equation is extremely important for linear system. In this example, I will first show the meaning of a matrix A^k and what happens when we take the matrix A to the infinity. Next I'll show how we can use this property to show that a certain equation is the solution to the discrete-time Lyapunov equation.

Let A $\in$ $\mathbb{R}^{\text{nxn}}$ and $M \in \mathbb{R}^{\text{nxn}}$ be given. Suppose that all the eigenvalues of A have magnitude strictly less than one.

(a) Show that A^k tends to zero as k tends to infinity.

Solution:

If we let A=$v\Lambda v^{-1}$ be the diagonization of of A. We know that for any function of A is the same as the function of the eigenvalue matrix such that:

$$f(A) = v f(\Lambda) v^{-1}$$

we know that $\Lambda = \begin{pmatrix} d_{11} & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & d_{22} & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & d_{33} & 0 & 0 & 0 & . & . & . \\ . & 0 & 0 & d_{44} & & & & & \\ . & & & & . & & & & \\ . & & & & & . & & & \\ . & & & & & . & . & & \\ & & & & & & . & . & \end{pmatrix}$

a function of the eigenvalue matrix would be

$$f(\Lambda) = \begin{pmatrix} f(d_{11}) & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & f(d_{22}) & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & f(d_{33}) & 0 & 0 & 0 & . & . & . \\ . & 0 & 0 & f(d_{44}) & & & & & \\ . & & & & . & & & & \\ . & & & & & . & & & \\ . & & & & & . & . & & \\ & & & & & & . & . & \end{pmatrix}$$

in our case we have $\lim_{k->\infty} d_{\text{ij}}{}^k = 0$

Since the eigenvalues are all strictly negative, the eigenvalue matrix would go towards zero as k approach infinity.

$$\Lambda^\infty = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & 0 & 0 & 0 & & & & & \\ . & & & & . & & & & \\ . & & & & & . & & & \\ . & & & & & . & . & & \\ & & & & & & . & . & \end{pmatrix}$$

now we have $f(A) = v[0]v^{-1} = 0$

From this we see that as long as the eigenvalue of A is strictly less than 1, the matrix cannot blowup.

(b) Show that the solution P of the discrete-time Lyapunov equation

P - APA^T = M is P = $\sum_{k=0}^{\infty} A^k M (A^T)^k$

Solution:

If we plug P into the Lyapunov equation we would get

$\sum_{k=0}^{\infty} A^k M (A^T)^k$ - A [$\sum_{k=0}^{\infty} A^k M (A^T)^k$] A^T = M

If can multiply the terms into the sum such that

$\sum_{k=0}^{\infty} A^k M (A^T)^k$ - [$\sum_{k=0}^{\infty} A^{k+1} M (A^T)^{k+1}$] = M

From this form we see that every term would cancel out except the zeroth term

$A^0 M (A^T)^0$ = M

Since A^0 is 1 we get

M = M

This shows that P is the solution for the discrete-time Lyapunov Equation.